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Invasion percolation on a random lattice

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Abstract. We use the technique of invasion percolation to compute the critical points and critical exponents of percolation on a random lattice which is the dual of a Voronoi network.

1. Introduction

The percolation problem was introduced by Broadbent and Hammersley [1] as a model for fluid flow in porous media. Since then many more applications of percolation theory have been found, in problems as diverse as the spread of disease in an orchard and the liquid-glass transition.

One of the most interesting discoveries which has been made about percolation theory is its relationship to the theory of critical phenomena. Both theories are concerned with the way in which physical quantities of the system under study vary as a critical point is approached. In percolation theory these physical quantities are directly related to the moments of the cluster distribution function which diverge as they approach the critical point with a power law behaviour governed by critical exponents characteristic of the universality class to which the system belongs. The same behaviour is displayed by the correlation functions in the theory of critical phenomena. Percolation theory is simpler than the theory of critical phenomena because no interactions are involved and it is a good means of obtaining intuition and, potentially, rigorous results which can be applied to the more complicated theory.

In this paper we study percolation on a random lattice and compute critical points and critical exponents. We are interested in comparing percolation on regular and random lattices. We hope that by demonstrating the universality of percolation on regular and random lattices we can give a basis for similar investigations on spin systems. Already numerical investigations have been performed for the Ising model [2] and the XY model [3] on two-dimensional random lattices, but because of the complex nature of these models it was not possible to obtain accurate values for the critical exponents or to demonstrate their universality with regular lattice models. Such studies are of particular interest for showing that the continuum limit of field theories on regular and random lattices is the same [2], a problem which is central to establishing the relevance of random lattice field theory [4-6].

Quite apart from these considerations it is of general interest to study percolation on a random lattice because many of the systems which exhibit percolation thresholds in nature are disordered systems (e.g. fluid flow in porous media, variable-range hopping in amorphous semiconductors) whilst by far the greater volume of work on percolation up to date has been restricted to studies on regular systems.

2. The random lattice

The random lattice which we will be discussing in this paper is of the type described by us in [3]. The sites of the lattice are points distributed at random in the plane. They are linked up to form triangles according to the criterion that the circumcircle of any triangle in the lattice contains no other site. It is known that this can be done in such a way that the triangles fill the plane with no overlap [4]. The dual of the random lattice is a lattice consisting of Voronoi polygons [7, 8]. The geometrical properties of these random lattices have been thoroughly researched [4–8].

Percolation in Voronoi networks has been discussed in papers by Winterfeld *et al* [9] and Jerauld *et al* [10] but they did not compute any critical exponents. We feel that we can extend their work by using a different method of computation, 'invasion percolation', which seems particularly suited to studies on random lattices.

3. Methods of simulating percolation

Usually in computer simulations of site percolation the sites of the lattice are assigned a random number between 0 and 1. A site is defined as being 'occupied' at the chosen percolation concentration p if it has been assigned a value less than p. Clusters of occupied sites are labelled and counted and cluster distributions are computed by averaging over several different assignments of random numbers to the lattice. The percolation concentration p is varied until the best estimate for the critical point is found.

An alternative method of investigation was introduced by Leath [11]. It involves growing a single cluster at a percolation concentration p and then averaging over several of these clusters to compute the probability distribution $P(n)$ of generating a cluster of size n. As the critical concentration is approached, the large-n tail of $P(n)$ grows so that the positive moments of the distribution diverge.

The algorithm for producing a cluster at concentration p is as follows.

(1) Choose a site to be a seed of the cluster.

(2) Assign random numbers between 0 and 1 to the nearest-neighbour sites of the seed. These sites form the boundary of the cluster to start with.

(3) Find the site on the boundary which has the smallest number, r, assigned to it.

(4) If $r < p$, then accept that site into the cluster and assign random numbers to any of its nearest neighbours which are empty, thereby adding them on to the boundary of the cluster. Repeat step (3).

(5) If $r > p$, then terminate the cluster.

Wilkinson and Barsony [12] considered a procedure very similar to the one above but in their version there was no percolation concentration p to be considered—at each stage the site on the boundary with the smallest random number was accepted into the cluster regardless. Hence the clusters were never constrained to terminate and could be grown to any desired size.

This process is called 'invasion percolation'. It was first introduced by Lenormand [13] and Chandler *et al* [14] in the context of a simulation of oil displacing water in a porous medium. Further studies [12, 15–17] have shown an interesting relationship between invasion percolation and ordinary percolation at threshold which allows invasion percolation to be used to study the critical behaviour of the ordinary percolation model.

4. Properties of invasion percolation

In invasion percolation a quantity called the 'acceptance profile' is computed. The acceptance profile $b_n(r)$ at value r and time n is defined by

$$b_n(r) = \frac{\langle \text{number of random numbers in } [r, r+\mathrm{d}r] \text{ accepted into cluster} \rangle_n}{\langle \text{number of random numbers in } [r, r+\mathrm{d}r] \text{ considered} \rangle_n}$$

where $\langle\ \ \rangle_n$ denotes an ensemble average over clusters of size n [12].

In Chayes *et al* [16] it is proved rigorously for invasion percolation on a square lattice in two dimensions that

$$\lim_{n\to\infty} b_n(r) = \begin{cases} 1 & \text{if } r < p_c \\ 0 & \text{if } r > p_c \end{cases}$$

where p_c is the critical concentration of the corresponding ordinary percolation model. Hence, for large n, the acceptance profile approaches a step function whose cut-off value coincides with p_c. We expect this result to apply equally well to invasion percolation on a two-dimensional random lattice.

For finite n the acceptance profile has a transition region around p_c, i.e. it is 'rounded off' in the manner illustrated in figure 1. It has been conjectured [12] that the rate of convergence of the acceptance profile to a step function follows a scaling behaviour governed by a critical exponent. More precisely, the quantities $B_1(n)$ and $B_2(n)$ defined by

$$B_1(n) = \int_0^{p_c} [1 - b_n(r)]\,\mathrm{d}r \qquad B_2(n) = \int_{p_c}^{1} b_n(r)\,\mathrm{d}r$$

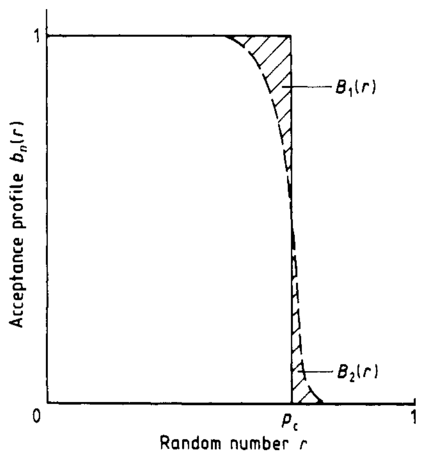

Figure 1. This figure illustrates the 'rounding off' of the acceptance profile due to finite-size effects in a typical invasion percolation calculation.

which represent the deviation of the acceptance profile from a step function are assumed to have a power law behaviour:

$$B_1(n) \sim b_1 n^{-1/\Delta} \qquad B_2(n) \sim b_2 n^{-1/\Delta} \qquad \text{as } n \to \infty$$

with a common exponent Δ.

Furthermore, it is conjectured that Δ should be identified with the gap exponent $\beta + \gamma$ of ordinary percolation. These conjectures are well supported by the results of computer simulations and by the exact solution of the invasion model on the Cayley tree [17].

Another unproven conjecture about the clusters grown in invasion percolation is that they have a fractal dimension, D, which is the same as that for the infinite cluster in ordinary percolation at threshhold. Within the accuracy that either value has been determined they do appear to agree.

The only rigorous results which have been shown for the invasion cluster in two dimensions are that, in the large-n limit,

(i) the surface to volume ratio has the value $(1-p_c)/p_c$, and

(ii) the invaded region has zero volume fraction [16].

These results are consistent with the conjecture that invasion percolation reproduces ordinary percolation at threshhold.

5. Discussion of results

In our work we followed the procedures of Wilkinson and Barsony [12] to measure the critical concentration p_c and the critical exponents $1/\Delta$ and $1/D$ for site and bond percolation on a random lattice. The invasion percolation technique works just as well for random as for regular lattices, whereas some of the other methods for measuring percolation quantities involve stepping through the lattice from site to site in a way which is special to regular lattices.

Our basic lattice was a periodic one of 10 000 sites. We started at a site near the centre of the lattice and grew clusters out into repetitions of the basic lattice. We grew clusters containing 3000 sites. The lower end of the range of n which we used in calculating critical exponents was given by $n_{\min} = 300$ (see [12]). All computations were done on the ICL DAP at Queen Mary College, London.

Our results are presented in table 1. As in [12], the errors quoted are one standard deviation statistical errors estimated by dividing the data into ten groups and observing the standard deviation between the ten sets of results.

The generally accepted values for the critical exponents $1/\Delta$ and $1/D$ are $1/\Delta = \frac{36}{91} = 0.3956$, $1/D = \frac{48}{91} = 0.5275$ [18-20]. It is seen that the values which we found for the critical exponents are the same as the expected ones within the quoted errors. Actually,

Table 1. Results of Monte Carlo simulations of invasion percolation for site and bond percolation on a random lattice.

Type	Number of clusters grown	p_c	$1/\Delta$	$1/D$
Site percolation	1000	0.502 ± 0.005	0.383 ± 0.042	0.534 ± 0.013
Bond percolation	1000	0.329 ± 0.002	0.411 ± 0.035	0.530 ± 0.014

the computed values are encouragingly close to the expected ones considering the modest size of the clusters which we grew.

The value which we found for the critical concentration p_c for site percolation on the random lattice is

$$0.502 \pm 0.005$$

which is consistent with the expectation [21] that a planar triangulated lattice will have a critical site percolation concentration of one half (although some exceptions have been constructed (see [22, 23]).

The value of p_c which we found for bond percolation is

$$0.329 \pm 0.002$$

in agreement with Jerauld *et al* [10]. This value is slightly lower than the corresponding value on a regular triangular lattice, presumably because the bond-averaged coordination number is slightly higher on the random lattice than on the regular one.

In conclusion, we have provided numerical evidence for the universality of percolation on regular and random lattices. This encourages us in our studies of statistical models and field theories on random lattices.

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